Computation of the Parallel-Plate Capacitor with Symmetrically Placed Unequal Plates

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Abstract —**The classical problem of the parallel-plate capacitor has been investigated by a number of authors, including Love [1] and Langton [2]; the latter gives good results that are still not complete as claimed [2]. In this paper, the constants occurring in the Schwarz–Christoffel equations are correctly related to the dimensions of the capacitor. The electric intensity is studied and four typical values are given along the central line of force in the general case of the parallel-plate capacitor with symmetrically placed unequal plates. For the symmetrical case of an equal plate, the complete field distribution is given by constructing the family of lines of force.**

I. Introduction

The problem of a parallel-plate capacitor with symmetrically placed unequal plates has a history of more than sixty years. Love [1] uses the standard procedure of conformal transformation to solve this problem, and Langton [2] has, in an excellent recent paper, attempted to give it a complete solution. Indeed, Langton has given good results useful for practical applications and yet his solution is still not complete and some of his remarks are not strictly correct.

II. The Problem

As shown in Fig. 1, the capacitor under study is of the parallel-plate type with symmetrically placed unequal plates (in this figure and in subsequent figures, we use solid lines to represent electrodes and dotted lines to represent flux lines). In Fig. 2, we take half of the whole field on the upper half of the z-plane to be investigated. Love and then Langton gave the following expression of the Schwarz–Christoffel equation to transform the boundaries in the z-plane into those in the z'-plane (Fig. 3)

$$\frac{dz}{dz'}=\frac{P(z'-C_1')(z'-C_2')}{\sqrt{(z'-x_1')(z'-x_2')(z'-x_3')(z'-x_4')}} \quad (1a)$$

where P is a constant to be determined.

In the z'-plane, the electrodes AC and EG are not of the same width. A second bilinear transformation

$$z'=z_0'+\frac{\lambda}{t+\nu} \quad (1b)$$

will transform the boundaries in the z'-plane into those in the t-plane with electrodes of the same width (Fig. 4). The

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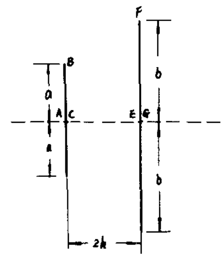

Fig. 1. The capacitor.

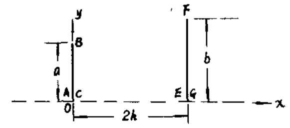

Fig. 2. The complex z-plane.

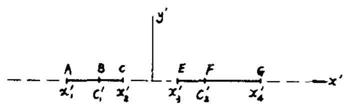

Fig. 3. The z'-plane.

real axis of the t-plane is folded to form a uniform field problem in the ω-plane by the following transformation (see Fig. 5):

$$t=\operatorname{sn}(\omega,k) \quad (1c)$$

where k is used in Fig. 4 such that $1/k$ is the distance OA.

In the ω-plane, we may write out the uniform field by a complex potential $W=U+jV$ and

$$W=\frac{1}{K}\omega \quad (2)$$

so that two electrodes are at $+1$ V and -1 V, where K is the complete elliptic integral of the first kind of modulus k

$$K=K(k). \quad (3)$$

The capacitance per unit length of the parallel-plate capacitor of Fig. 5 with a uniform field inside is

$$C'=\epsilon\frac{K'}{2K} \quad (4a)$$

where ϵ is the permittivity of the medium. By the principle of conservation of capacitance of the capacitor in conformal transformations, the capacitance per unit length of the

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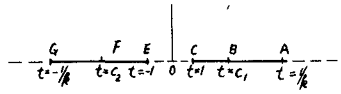

Fig. 4. The t-plane.

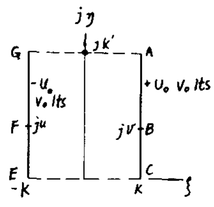

Fig. 5. The ω-plane, $\omega = \xi + j\eta$.

capacitor shown in Fig. 1 is just twice the value of (4a)

$$C = 2C' = \epsilon \frac{K'}{K} \tag{4b}$$

and the field distribution inside and around the capacitor of Fig. 1 can be mapped by the potential lines and the flux lines given by (2)

$$W = U(x, y) + jV(x, y) = \frac{1}{K}\omega(z)$$

and we take $U(x, y)$ as the potential function, while $V(x, y)$ is then the flux function.

The problem remaining to be solved now is how we are going to relate the dimensions a, b, and h in Fig. 1 to the parameter k of (3), which again appears in (2) and (4b) implicitly.

Langton [2] obtains the parameter k by first prescribing the a', b', and h' of Fig. 3, where

$$a' = X_2' - X_1'$$
$$b' = X_4' - X_3'$$
$$h' = X_3' - X_2'$$

then gives k as the root less than 1 of the following equation:

$$k^2 - 2k\left\{1 + \frac{2a'b'}{h'(a' + b' + h')}\right\} + 1 = 0. \tag{5a}$$

Having found k from (5a), the points C_1 and C_2 in Fig. 4 can be solved from the following relations, which are obtained from Love's results [1] by Langton [2]:

$$C_1 + C_2 = -\nu \frac{2E' - (1 + k^2)K'}{E' - k^2\nu^2 K'} \tag{5b}$$

$$C_1 C_2 = \frac{E'\nu^2 - K'}{E' - \nu^2 k^2 K'} \tag{5c}$$

provided ν is known. One form of this constant ν in the bilinear transformation (1b) is given by Langton [2] by prescribing a', b', and h' as follows:

$$\nu = \frac{h' - h'k + 2b'}{2b'k - h' + h'k} = \frac{h'k - 2a' - h'}{2a'k - h' + h'k}. \tag{5d}$$

Langton determines the constants in the transformation from the z-plane of Fig. 2 to the ω-plane of Fig. 5 to give

$$z = h - \frac{2hk'}{\pi}\left\{ Z(\omega) + \frac{\pi\omega}{2KK'} + \frac{\mathrm{cn}\omega\mathrm{dn}\omega}{\mathrm{sn}\omega + \nu}\right\}. \tag{5e}$$

However, Langton did not relate the lengths a', b', and h' to the actual dimensions of Fig. 1 and instead said erroneously that since the transformation from the z- to the z'-plane is conformal, then $a/a' = b/b'$ [2, eq. (23)], and furthermore implied in his example that $a/a' = b/b' = h/h'$, which is also not strictly correct.

We are now to do the work of relating the parameters X_1', X_2', X_3', and X_4' to a, b, and h to complete this problem of the parallel-plate capacitor with symmetrically placed unequal plates.

III. The Integration of (1)

Langton [2] remarked that (1) cannot be integrated. This is not true. In fact, (1) can be integrated by [3, eqs. 250]. Now we proceed to integrate (1) to relate the constants x_1', x_2', x_3', and x_4' to the dimensions of the capacitor a, b, and h of Figs. 1 and 2. To make use of [3, eqs. 252.11 and 253.11], we put (1a) into the following form (where $j = \sqrt{-1}$):

$$\frac{dz}{dz'} = -P \frac{j(z' - C_1')(z' - C_2')}{\sqrt{(X_4' - z')(X_3' - z')(X_2' - z')(z' - X_1')}}. \tag{6}$$

Integrating both sides from points A to C of Figs. 2 and 3, we have

$$0 = -jP\int_{X_1'}^{X_2'} \frac{z'^2 - (C_1' + C_2')z' + C_1'C_2'}{\sqrt{(X_4' - z')(X_3' - z')(X_2' - z')(z' - X_1')}}. \tag{6a}$$

We then have one equation relating $(C_1 + C_2)$ and $C_1'C_2'$. The integrals in (6a) are the same as those to follow in (7) and will be evaluated there.

To make use of point B, [3, eq. 253]

$$z = -jP\left[\int_y^{X_2'} - \int_{X_1'}^{X_2'}\left\{\frac{C_1'C_2' - (C_1' + C_2')z' + z'^2}{\sqrt{(X_4' - z')(X_3' - z')(X_2' - z')(z' - X_1')}}dz'\right\}\right] \tag{7}$$

for

$$X_1' \leqslant y < X_2' < X_3' < X_4'$$

so that $z = 0$ when $z' = y = X_1'$. The integral (7) will be given in terms of (from [3, eq. 253.11], m is the power of z'

in the numerator of (7))

$$gX_2'^m \int_0^{u_1} \left(\frac{1-\alpha_1^2 \mathrm{sn}^2 u}{1-\alpha^2 \mathrm{sn}^2 u} \right)^m du = X_2'^m Z_m \tag{8}$$

where

$$Z_0 = u = F(\varphi, k) = V_0$$

$$Z_1 = \tfrac{1}{2}\left[(\alpha^2 - \alpha'^2)V_1 + \alpha'^2 V_0\right]$$

$$Z_2 = \frac{1}{\alpha 4}\left[\alpha'^4 V_0 + 2\alpha'^2(\alpha^2 - \alpha'^2)V_1 + (\alpha^2 - \alpha'^2)V_2\right]$$

$$V_1 = \pi(\varphi, \alpha^2, k)$$

$$V_2 = \frac{1}{2(\alpha^2 - 1)(k^2 - \alpha^2)} \cdot \left[\alpha^2 E(u) + (k^2 - \alpha^2)u + (2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)V_1 - \frac{\alpha^4 \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u}{1 - \alpha^2 \mathrm{sn}^2 u}\right] \tag{9}$$

and

$$g = \frac{1}{\sqrt{(X_4' - X_2')(X_3' - X_1')}}$$

$$k^2 = \frac{(X_4' - X_3')(X_2' - X_1')}{(X_4' - X_2')(X_3' - X_1')}$$

$$k^2 < \alpha^2 = \frac{X_2' - X_1'}{X_3' - X_1'} < 1$$

$$\alpha'^2 = \frac{X_3'}{X_2'} \frac{X_2' - X_1'}{X_3' - X_1'} \tag{10}$$

$$\phi = \mathrm{cm}\, u = \sin^{-1}\sqrt{\frac{(X_3' - X_1')(X_2' - y)}{(X_2' - X_1')(X_3' - y)}}, \qquad \mathrm{sn}\, u = \sin\phi \tag{11}$$

At point B of Figs. 2 and 3

$$z = ja \quad z' = C_1'$$

and from (11), for $y = X_1'$, $\sin\phi = \mathrm{sn} u = 1$; therefore

$$u = K \quad \phi = \pi/2$$

and we write

$$\phi_3 = \mathrm{am}\, U_3 = \sin^{-1}\sqrt{\frac{(X_3' - X_1')(X_2' - C_1')}{(X_2' - X_1')(X_3' - C_1')}} \tag{12}$$

and add a subscript to the quantities in (10) to signify that [3, eqs. 253] have been used. Equation (7) then goes into

$$a = -gP\Big[C_1'C_2'\Big\{Z_0/\phi_3, \alpha_3, k_3) - Z_0\Big(\frac{\pi}{2}, \alpha_3, k_3\Big)\Big\} - (C_1' + C_2')X_2'\Big\{Z_1(\phi_3, \alpha_3, k_3) - Z_1\Big(\frac{\pi}{2}, \alpha_3, k_3\Big)\Big\} + X_2'^2\Big\{Z_2(\phi_3, \alpha_3, k_3) - Z_2\Big(\frac{\pi}{2}, \alpha_3, k_3\Big)\Big\}\Big]$$

where Z_0, Z_1, and Z_2 are given in (9) when ϕ, α, and k have been written as ϕ_3, α_3, and k_3 and are the same as (12). More explicitly, we write out

$$a = gP\Big[C_1'C_2'(u_3 - k_3) - (C_1' + C_2')\Big\{-h'V_1(\phi_3, \alpha_3, k_3) + h'V_1\Big(\frac{\pi}{2}, \alpha_3, k_3\Big) + X_3'(u_3 - k_3)\Big\} + \Big\{X_3'^2(u_3 - k_3) - 2h'X_3'V_1(\phi_3, \alpha_3, k_3) + 2h'X_3'V_1\Big(\frac{\pi}{2}, \alpha_3, k_3\Big) + h'^2V_2(\phi_3, \alpha_3, k_3) - h'^2V_2\Big(\frac{\pi}{2}, \alpha_3, k_3\Big)\Big\}\Big] \tag{13}$$

where we have adopted the notation of Langton

$$h' = X_3' - X_2'$$
$$a' = X_2' - X_1'$$
$$b' = X_4' - X_3'. \tag{14}$$

We also write the complete elliptic integral of the first kind as

$$K_3 = K(k_3)$$

and gP is a constant to be evaluated. V_0, V_1, and V_2 can be found from (9).

To integrate from points C to E of Figs. 2 and 3, we take [3, eq. 254] and put (1a) into the following form:

$$dz = P\frac{(z' - C_1')(z' - C_2')}{\sqrt{(X_4' - z')(X_3' - z')(z' - X_2')(z' - X_1')}}dz'.$$

Integrating, we find

$$2h = P\int_{X_2'}^{X_3'} \frac{z'^2 - (C_1' + C_2')z' + C_1'C_2'}{\sqrt{(X_4' - z')(X_3' - z')(z' - X_2')(z' - X_1')}}. \tag{15}$$

Next, we integrate from points E to G; by [3, eq. 256] we have again, from (1b)

$$0 = \frac{P}{j}\int_{X_3'}^{X_4'} \frac{z'^2 - (C_1' + C_2')z' + C_1'C_2'}{\sqrt{(X_1' - z')(z' - X_3')(z' - X_2')(z' - X_2')}} \tag{16}$$

so that we have another equation relating $C_1' + C_2'$ and $C_1'C_2'$.

At the point F of Figs. 1 and 2, we make use of [3, eq. 257.11], and write

$$z = -jP\left[\int_y^{X_4'} - \int_{X_3'}^{X_4'}\left\{\frac{z'^2 - (C_1' + C_2')z' + C_1'C_2'}{\sqrt{(X_4' - z')(z' - X_3')(z' - X_2')(z - X_1')}}dz'\right]\right. + 2h \tag{17}$$

so that when $z=2h+j0$, $z'=y=X_3'$, then $y=C_2'$ gives

$$z=2h+jb.$$

We have from (17)

$$b=-P\left[\int_{C_2'}^{X_4'}-\int_{X_3'}^{X_4'}\left\{\frac{z'^2-(C_1'+C_2')z'+C_1'C_2'}{\sqrt{(X_4'-z')(z'-X_3')(z'-X_2')(z'-X_1')}}\right\}\right]. \tag{18}$$

Then, proceeding as we have done from (7) to (13), we write

$$k_7^2=\frac{a'b'}{(a'+b')(b'+h')}$$

$$\alpha_7'^2=-\frac{X_1'}{X_4'}\frac{b'}{b'+h'}$$

$$\phi_7=\mathrm{am}\,U_7=\sin^{-1}\sqrt{\left(\frac{a'+h'}{b'}\frac{X_4'-C_2'}{C_2'-X_1'}\right)}$$

$$X_1'<X_2'<X_3'\leqslant y<X_4',$$

$$\phi_7=\pi/2 \text{ for } y=X_3' \quad K_7=K(k_7)=K_3.$$

Then (18) gives

$$\begin{aligned} b=-gP\Big[&C_1'C_2'(U_7-K)-(C_1'+C_2')\Big\{(a'+b'+h')\\ &\cdot V_1(\Phi_7,\alpha_7,k_2)\\ &-(a'+b'+h')V_1\left(\frac{\pi}{2},\alpha_7,k_2\right)\\ &+X_1'(U_7-K)\Big\}+\Big\{X_1'^2(U_7-K)\\ &+2(a'+b'+h')\\ &\cdot X_1'V_1(\Phi_7,\alpha_7,k_3)-2(a'+b'+h')X_1'V_1\left(\frac{\pi}{2},\alpha_7,k_3\right)\\ &+(h'+a'+b')^2V_2(\Phi_7,\alpha_7,k_3)\\ &-(h'+a'+b')V_2(\Phi_7,\alpha_7,k_3)\Big\}\Big]. \end{aligned} \tag{19}$$

Now we have two equations (6b) and (16) to solve for $C_1'C_2'$ and $C_1'+C_2'$, one equation (15) to solve for the constant P, and (13) and (19) determine the values a/h and b/h completely.

IV. Capacitor Plate Dimensions from the Second Integration of (1)

Actually, (1) has been integrated by Love [1, eq. (5e)], from which the dimensions of the capacitor of Fig. 1 can be correctly determined as follows.

From Figs. 2 and 5, at points B and F, we have, respectively,

$$z=ja \quad \omega=K+jv \quad t=C_1 \tag{20a}$$

and

$$z=jb+2h \quad \omega=-K+ju \quad t=C_2. \tag{20b}$$

So by (1c) we have

$$C_1=\mathrm{sn}(K+jv,k)=\mathrm{dn}(v,k'). \tag{20c}$$

By [3, eq. 125.0], recalling $\mathrm{cn}\,K=0$ and $\mathrm{dn}\,K=k'$, and by (1c) again we have

$$C_2=-\mathrm{dn}(u,k'). \tag{20d}$$

Putting (20a) and (20b) into (5e)

$$\frac{a}{h}=\frac{2K'}{\pi}\left\{Z(\nu,k')-k'^2\nu\frac{\mathrm{cn}(\nu,k')\mathrm{sn}(\nu,k')}{1+\nu\,\mathrm{dn}(\nu,k')}\right\} \tag{21a}$$

$$\frac{b}{h}=\frac{2K'}{\pi}\left\{Z(u,k')+k'^2\nu\frac{\mathrm{cn}(u,k')\mathrm{sn}(u,k')}{1-\nu\,\mathrm{dn}(u,k')}\right\}. \tag{21b}$$

We thus have related prescribed parameters a', b', and h' of Figs. 3 to the capacitor dimensions a, b, and h, since k is then given by (5a) and ν by (5b), while (20c) and (20d) will yield v and u.

From (21a) and (21b), we can find the ratio of plate widths

$$\frac{a}{b}=\frac{z(v,k')-\dfrac{k'^2\nu\,\mathrm{cn}(v,k')\mathrm{sn}(v,k')}{1+\nu\,\mathrm{dn}(v,k')}}{z(u,k')-\dfrac{k'^2\nu\,\mathrm{cn}(u,k')\mathrm{sn}(u,k')}{1-\nu\,\mathrm{dn}(u,k')}} \tag{21c}$$

where

$$\mathrm{dn}(u,k')=-\frac{1}{C_2} \quad \mathrm{dn}(v,k')=+\frac{1}{C_1}$$

$$\mathrm{sn}(u,k')=\frac{1}{k}\sqrt{\left(1-\frac{1}{C_2^2}\right)}$$

$$\mathrm{cn}(u,k')=\sqrt{\frac{(1-k^2C_2^2)}{C_2k'}}$$

$$\mathrm{sn}(v,k')=\frac{1}{k'}\sqrt{\left(1-\frac{1}{C_1^2}\right)}$$

$$\mathrm{cn}(v,k')=\sqrt{\frac{(1-k^2C_1^2)}{C_1k'}}, \qquad 0\leqslant v\leqslant K', 0\leqslant u\leqslant K'.$$

According to [2, eq. (23)], we would have, instead of (21c)

$$\frac{a}{b}=\frac{a'}{b'}=\frac{(k\nu+1)(\nu+1)}{(k\nu-1)(\nu-1)}$$

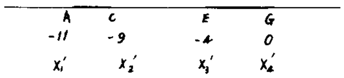

Fig. 6. The z' parameters.

which is not true, as demonstrated by the numerical example below.

For the prescribed a/h and b/h, we have theoretically four equations ((5b), (5c), (21a), and (21b)) to solve for four parameters k, ν, u, and v so that we have the required k for (4a). Actually, it is difficult to solve these four equations given only a/h and b/h, so it is expedient to start with prescribed a'/h' and b'/h' approximately to give the required ratio of a/b and to proceed further to get the correct values of a/h and b/h.

Before we give the numerical example, we first transform (5e) into the following form by [3, eq. 140.01]:

$$z = h - \frac{2kK'}{\pi}\left\{E(\omega) - \frac{K'-E'}{K'}\omega + \frac{\operatorname{cn}\omega\operatorname{dn}\omega}{\operatorname{sn}\omega+\nu}\right\} \tag{21d}$$

so that we can make use of some good tables of elliptic function with complex argument, such as [4]. By putting the points B and F for $\omega = K + j\nu$ for $z = ja$ and $\omega = -K + ju$ for $z = 2h - jb$, we find

$$\frac{a}{h} = -\frac{2kK'}{\pi}\left\{(K'-E')\left(\frac{E_t}{K'-E'} - \frac{u}{K'}\right) - k'\frac{\operatorname{sn}(v,k')(\operatorname{cn}(v,k')}{\operatorname{dm}(v,k')\{1-\nu\operatorname{dn}(v,k')\}}\right\} \tag{21e}$$

$$\frac{b}{h} = -\frac{2kK'}{\pi}\left\{(K'-E')\left(\frac{E_i}{K'-E'} - \frac{u}{K'}\right) + k'^2\frac{\operatorname{sn}(u,k')\operatorname{cn}(u,k')}{\operatorname{dn}(u,k')\{1-\nu\operatorname{dn}(u,k')\}}\right\} \tag{21f}$$

so that when $\nu \to \infty$ as $a \to b$, only one term remains on the right-hand sides of (21e and f).

To demonstrate now that a/b has to be worked out, and not simply being given by a'/b', we will work on the example in Langton's paper [2]. The z' parameters are prescribed in Fig. 6, so

$$a' = 2$$
$$h' = 5$$
$$b' = 4$$

and (5a) gives

$$k = 0.475$$

and (5b) gives

$$\nu = 9.$$

From (20c) and (20d), we find that

$$v = 1.19043$$
$$u = 1.25153$$

and then

$$Z(v,k') = 0.256141$$
$$Z(u,k') = 0.248735.$$

Equations (21a) and (21b) give

$$\frac{a}{h} = 0.266003$$
$$\frac{b}{h} = 0.524377.$$

Therefore

$$\frac{b}{a} = 1.97132$$

which is not 2 as was supposed by Langton [2]; moreover, the ratios a/h and b/h are far from 0.4 and 0.8, respectively.

So we can see that we may start with a prescribed dimension of the transformed dimensions such as those in Fig. 6, and then work out k and v by (5a) and (5b), to be inserted in 21(a) and (21b) to find a/h and b/h and then a/b by (21c).

V. The Electric Intensity Distribution

For industrial heating applications and in antenna work, we wish to know the electric-field intensity distribution inside the capacitor and in its neighborhood. From the complex potential W of (2), we have

$$\frac{dW}{dz} = \frac{dW}{dx} = \frac{\partial U}{\partial x} + j\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x} - j\frac{\partial U}{\partial y} = -E_x + jE_y \tag{22a}$$

or

$$-E_x + jE_y = \frac{dW}{dw}\frac{dw}{dz} = \frac{\frac{1}{K}}{\frac{dz}{dw}}. \tag{22b}$$

dz/dw is given by operating on Love's result quoted in (5e) to obtain

$$\frac{dz}{dw} = -\frac{2h}{\pi}(E' - k'\nu^2K')\frac{(t-C_1)(t-C_2)}{(t+\nu)^2},$$
$$t = \operatorname{sn}(\omega, k). \tag{22c}$$

Consequently, we can write

$$-E_x + jE_y = \frac{U_0}{\pi}\frac{\pi}{2h}\frac{1}{E' - k^2\nu^2K'}\frac{(t+\nu)^2}{(t-C_1)(t-C_2)} \tag{23a}$$

for a potential difference of $2V_0$ between the capacitor plates of Fig. 1.

Along the center line CE of Fig. 1, t is real

$$-1 \leqslant t \leqslant 1.$$

For heating applications, we wish to obtain as nearly a uniform electric-field intensity distribution as possible, so we want the following four values of $|E| = |-E_x + jE_y|$ as nearly equal as possible at $t = 1, 0, -1$ and at t for which

$$\frac{d|E|}{dt} = 0 \tag{23b}$$

which gives

$$t = \frac{2C_1C_2 + \nu(C_1 + C_2)}{C_1 + C_2 + 2\nu}. \tag{23c}$$

By (5b) and (5c), this is the same as

$$t = \frac{(1+k^2)\nu^2 - 2}{1 + k^2 - 2k^2\nu^2}\frac{1}{\nu}. \tag{23d}$$

At this value of t, the value $|E|$ is a minimum

$$|E_{\min}| = \frac{4V_0}{h}\frac{\pi}{2K} \cdot \left| \frac{(V^2-1)(k^2V^2-1)K'}{4E'K'(1-V^2)(1-K^2V^2) + V^2K'^2(1-k^2)^2} \right| = \frac{U_0}{h}\frac{\pi}{2KE'}\frac{1}{1 + \dfrac{V^2}{4}\dfrac{K'}{E'}\dfrac{(1-k^2)^2}{(V^2-1)(1-v^2k^2)}}. \tag{24a}$$

To obtain a uniform field intensity along the center line CE, we wish to make (24a) as nearly equal to $|E|$ at $t = \pm 1$, 0 as possible, where, at $t = \pm 1$, we have

$$t = -1: \qquad |E_1| = \frac{U_0}{h}\frac{\dfrac{\pi}{2KE'}}{1 - \dfrac{K'}{E'}\dfrac{(1-\nu)(1-\nu k^2)}{(\nu+1)^2}} \tag{24b}$$

and

$$t = 1: \qquad |E_2| = \frac{U_0}{h}\frac{\dfrac{\pi}{2KE'}}{1 - \dfrac{K'}{E'}\dfrac{(1+\nu)(\nu k^2+1)}{(\nu-1)^2}} \tag{24c}$$

where C_1C_2 and $C_1 + C_2$ are given by solving (5b) and (5c), so we obtain

$$|E_1| = \frac{U_0}{h}\frac{\dfrac{\pi}{2KE'}}{1 + \dfrac{K'}{E'}\dfrac{\nu k^2 - 1}{\nu - 1}}\frac{1 - \nu^2kK'/E'}{1 - \nu^2k^2K'/E'} \tag{25a}$$

$$|E_2| = \frac{U_0}{h}\frac{\dfrac{\pi}{2KE'}}{1 + \dfrac{K'}{E'}\dfrac{\nu k^2 + 1}{\nu + 1}}\frac{1 - \nu^2kK'/E'}{1 - \nu^2k^2K'/E'} \tag{25b}$$

while for $t = 0$, (23a) gives

$$t = 0: \qquad |E_0| = \frac{U_0}{h}\frac{\dfrac{\pi}{2}\Big/KE'}{1 - k^2\nu^2K'E'}\frac{\nu^2}{C_1C_2} = \frac{U_0}{h}\frac{\pi}{2KE'}\frac{1}{1 - \dfrac{K'}{\nu^2E'}}. \tag{25c}$$

We have found the field intensities at four points, C, E, their midpoint, and another point given by (23d). By choosing the appropriate k and v from their permissible sets of k and v obtained from solutions of (5b), (5c), and (5d), we can obtain a nearly uniform field intensity distribution along the whole CE of Fig. 1 or a better distribution over a portion of the line segment CE.

To study the electric-field intensity inside and outside the capacitor, we make use of the fact that the equipotentials and the flux lines between the electrodes of Fig. 1 have been transformed into vertical and horizontal lines in the uniform field in the ω-plane of Fig. 5 and (2) and (5e). Thus, for $U_0 = 1$ V, we have

$$U(x, y) = \frac{1}{K}\xi \tag{26a}$$

$$V(x, y) = \frac{1}{K'}\eta\frac{K'}{K} \tag{26b}$$

so that the total flux passing through a line joining any two points in the field of Fig. 1 equals the product of the permittivity ϵ by the difference in the value of the stream function V at the two points [5]. The potential function U representing the equipotentials and the stream function V representing the lines of force are both straight lines, so for $V(x, y) = V_1$, (26b) gives

$$\eta = K'\frac{V_1}{K'/K}$$

and along this line of force $V(x, y) = V_1$

$$\omega = \xi + j\frac{V_1}{K'/K}K'. \tag{26c}$$

On substituting this ω into (5e), we obtain an equation of a line of force in Fig. 1 with ξ as the parameter as ξ changes from $-K$ to K.

Since the completely symmetrical case of $a = b$ in Fig. 1 has received more attention than the more general case, we will give more details on this case in the following section.

VI. The Symmetrical Case of a Capacitor of Equal Plates

For the case $a = b$ in Fig. 1, (21a) and (21b) are the same provided $\nu \to \infty$. Then (5e) goes into

$$z = h - \frac{2K'h}{\pi}\left[Z(\omega) + \frac{\pi\omega}{2KK'}\right] \tag{27a}$$

or

$$z = h - \frac{2K'h}{\pi}\left[E(\omega) - \frac{K' - E'}{K'}\omega\right] \tag{27b}$$

from (21d) for $\nu \to \infty$. From Fig. 4, $\omega = K + jV$, $z = ja$, (27a) gives, by [3, eq. 143.01]

$$\frac{a}{h} = +\frac{2K'}{\pi}\left[Z(u,k') - k'^2 \frac{\operatorname{sn}(v,k')\operatorname{cn}(v,k')}{\operatorname{dn}(v,k')}\right] \tag{28a}$$

while (27b) yields

$$\frac{a}{h} = -\frac{2K'}{\pi}(K'-E')\left\{\frac{E_i(v,k)}{K'-E'} - \frac{v}{K'}\right\} \tag{28b}$$

by the known relations [3, eq. 110.10] [4, pt. IV])

$$E(K+jv,k) = E(k) + jE_i(v,k)$$

and

$$EK' + E'K - KK' = \pi/2.$$

The values C_1 and C_2 of Fig. 4 will become

$$C_1 = -C_2 = C = \frac{1}{k}\sqrt{\frac{E'}{K'}}. \tag{28c}$$

So the electric-field intensity will be given by (23a) as

$$|E| = \frac{U_0}{h}\frac{\pi/2}{KE'}\frac{1}{\left|1 - k^2\dfrac{K'}{E'}\operatorname{sn}^2\omega\right|} \tag{28d}$$

which has the minimum value

$$|E_{\min}| = \frac{U_0}{h}\frac{\pi/2}{KE'} \tag{28e}$$

for

$$t = \operatorname{sn}(\omega,k) = 0$$

and

$$|E_1| = \frac{|E_{\min}|}{1 - k^2K'/E'} \tag{28f}$$

by (25a). Now the value of u or v in Fig. 4 is

$$\operatorname{sn}(K+jv) = C$$

so

$$\operatorname{dn}(v,k') = \frac{1}{C} = k\sqrt{\frac{K'}{E'}} \tag{28g}$$

by (28c). With all this information, we can easily transform the field in the ω-plane (26a) and (26b) back to the z-plane for $a = b$ by means of (27a) or (27b). In Table I, given such a transformation, we take (27b) and rewrite it as follows:

$$\begin{aligned}
\frac{Z}{h} &= -\frac{z-h}{h} \\
&= \frac{2K'}{\pi}\left\{Er(\xi,\eta) + jEi(\xi,\eta) - \frac{K'-E'}{K'}\xi - j\frac{K'-E'}{K'}\eta\right\} \\
&= \frac{2K'E}{\pi}\left\{\frac{Er(\xi,\eta)}{E} - \frac{K'-E'}{K'}\frac{K}{E}\frac{\xi}{K}\right\} \\
&\quad + j\frac{2K'(K'-E')}{\pi}\left\{\frac{Ei(\xi,\eta)}{K'-E'} - \frac{\eta}{K'}\right\} \\
&= \frac{x}{h} + j\frac{y}{h}.
\end{aligned} \tag{29}$$

The quantities K, K', E, E', $E_r' = Er(\xi,\eta)/E$, $Ei' = Ei(\xi,\eta)/(K'-E')$ can be read from [3, pt. IV]. An example is shown in Fig. 7, indicating that the boundary line of force originating from the tip of one of the capacitor plates and entering orthogonally onto the ground plane $x = 0$, for the case of $a/h = 0.5386$ when $k = \sin 20° = 0.34202$, with $\nu/K' = 0.5832$. In this manner, we can construct the complete families of the equipotentials and the lines of force in the transverse section of the two-dimensional capacitor.

TABLE I
LOCI OF THE LINE OF FORCE

x/h	0	.2696	.5066	.6497	.8232	.9585	.9852	.9974	.9992	1.0000
y/h	1.3867	1.3462	1.2492	1.1186	.9771	.7161	.6480	.5868	.5506	.5386
η = 0.5K'										
x/h	0	.2967	.5309	.7218	.8533	.9811	1.0025	1.0088	1.0063	1.0000
y/h	1.4433	1.4057	1.2996	1.1582	1.0056	.7473	.6560	.5917	.5539	.5412
η = 0.7K'										
x/h	0	.4717	.8253	1.0474	1.1531	1.1723	1.1392	1.0962	1.0489	1.0000
y/h	2.0201	1.9278	1.6930	1.4101	1.1427	.9769	.6376	.5585	.5134	.4988
η = 0.8K'										
x/h	0	.9520	1.4964	1.6567	1.6247	1.4687	1.2967	1.1922	1.0943	1.0000
y/h	3.1208	2.8202	2.2036	1.5869	1.1471	.6520	.5238	.4465	.4020	.3881
η = 0.9K'										
x/h	0	2.9305	3.1203	2.6495	2.2124	1.5300	1.4337	1.2714	1.1307	1.0000
y/h	6.3366	4.8182	2.3539	1.3107	.8067	.5419	.3912	.2498	.2232	.2130
η = 0.1K'										
x/h	0	.1243	.2323	.3443	.5147	.5407	.7430	.8312	.9162	1.0000
y/h	.1793	.1781	.1734	.1686	.1619	.1468	.1401	.1350	.1320	.1308
η = 0.2K'										
x/y	0	.1306	.2439	.3600	.4696	.6681	.7576	.8416	.9215	1.0000
y/h	.3649	.3606	.3536	.3408	.3249	.2913	.2770	.2657	.2584	.2559
η = 0.3K'										
x/h	0	0.1430	.2673	.3914	.5054	.7012	.7848	.8608	.9315	1.0000
y/h	.5649	.5594	.5438	.5201	.4911	.4199	.4059	.3859	.3737	.3700
η = 0.4K'										
x/h	0	.1661	.3098	.4473	.5678	.7571	.8300	.8924	.9478	1.0000
y/h	.7924	.7824	.7539	.7117	.6615	.5607	.5193	.4885	.4692	.4628
η = 0.5K'										
x/h	0	.0208	.3866	.5428	.8171	.8460	.9002	.9405	.9723	1.0000
y/h	1.0695	1.0510	0.9986	.9923	.8355	.6717	.6085	.5621	.5340	.5245

Boundary line: $\eta = V_1$, $V/K' = 0.58324$, $k = \sin 20°$, $a/h = 0.5386$.

In Table II, we tabulate some of the important data of the symmetrical parallel-plate capacitor: $1/C$, which appears in (28d); the field $|E|$ is more uniform the smaller $1/C$ is in the neighborhood of $\omega = 0$. This is the useful region for industrial heating applications. ν/K': the value of total flux bounded by the stream function through the tip of the plate of the capacitor, this is the fraction of the total flux under one half of the capacitor plate, so $1 - v/k'$ is the amount of flux originating from the upper side plate. z_b/h is the height where the boundary line of force meets the ground plane: it is a measure of the degree of confinement of the field by the capacitor plates, and its value is found by putting $\omega = 0 + jv$ into (27a) or (27b)

$$z_b = -\frac{z-h}{h} = \frac{2K'}{\pi}\left\{Z(v,k') - \frac{k'^2\operatorname{cn}(v,k')\operatorname{sn}(v,k')}{\operatorname{dn}(v,k')}\right\} \tag{30a}$$

by [3, eq. 143.01], or

$$z_b = \frac{2K'}{\pi}\left[(K'-E')\left\{E_i'(0,v) - \frac{v}{K'}\right\}\right] \tag{30b}$$

by [4, pt. IV]. The last two columns are from (28e) and (28f) to show the uniformity of the field intensity in the neighborhood of the midpoint of the inner side of the plate.

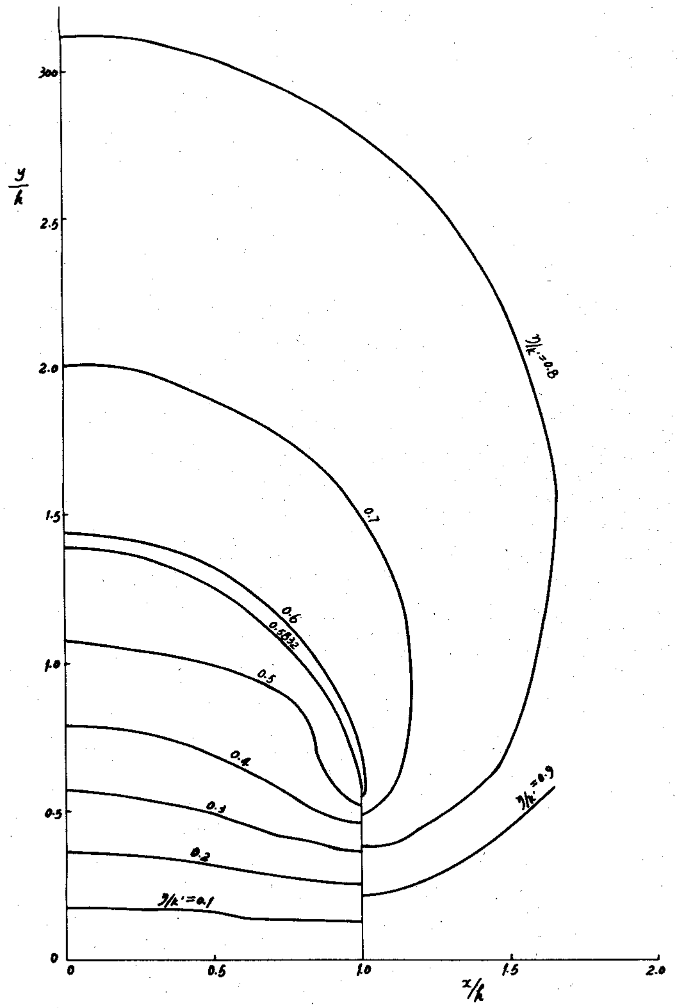

Fig. 7. A field line plot, $a = b$.

TABLE II
SYMMETRICAL PARALLEL-PLATE CAPACITOR OF EQUAL WIDTH

$\alpha = \sin^{-1}k$	$dn(\nu, k') = 1/C$	ν/k'	a/h	z_b/h	z_b/h	$\frac{E_{min}}{U_0/h}$	$\frac{E_a}{U_0/h}$
	(28g)	(28g)	(28a, b)	(30a, b)	(ratio)	(28e)	(28f)
1°	.0406713	.7322	2.1320	3.004	1.4090	.99917	1.00083
5°	.169544	.6674	1.2069	2.0651	1.7111	.98561	1.01478
10°	.302355	.6285	.8862	1.7105	1.9301	.95412	1.05325
15°	.415047	.6006	.6769	1.4849	2.1936	.91312	1.103154
20°	.511826	.5832	.5386	1.3807	2.5035	.81908	1.109812
25°	.595245	.5723	.4300	1.3168	3.0624		
30°	.667215	.5604	.3205	1.2558	3.9185	.76942	1.38679
35°	.729264	.5452	.2753	1.1889	4.3180		
40°	.782670	.5361	.2200	1.1478	5.2171	.72924	1.88225
45°	.828473	.5283	.1717	1.1137	6.4867		
50°	.867543	.5219	.1233	1.0848	8.8000	.58252	2.35486
55°	.900592	.5164	.09111	1.0643	11.6810		
60°	.928204	.5118				.49636	3.885512
65°	.950852	.5076					
70°	.968909	.5054				.41159	6.72363
75°	.982659	.5029					
80°	.992346	.5023				.31954	20.94413
85°	.998094	.5000				.26216	68.8443
89°	.999924	.5000				.18400	1210.53

VII. CONCLUSIONS

Conformal transformations are very powerful tools for solving two-dimensional boundary value problems thanks to the availability of modern computers, because good tables of mathematical functions can be obtained to facilitate the numerical computation of results valuable for practical applications. Classical problems having a history of many years may find new applications and, therefore, deserve our critical study. Langton [2] endeavors to give a complete solution for a capacitor with symmetrically placed plates of finite unequal breadths. Good results have been achieved by Langton, but his solution is still not complete. He stated that equation (1) of his paper cannot be integrated; actually, it can be done by formulas in Byrd and Friedman's celebrated *Handbook* [3]. Langton misrepresents the property of conformal mapping by his equation (23). This equation should be replaced by (21a), (21b), and (21c) of this paper. The statement that values of elliptic integrals for complex arguments have not been tabulated is not true; we use one of the tables in [4] to calculate the whole field distribution of the symmetrical parallel-plate capacitor of equal plates (Fig. 7 of this paper). We will give more numerical results for practical application in subsequent work, as this classical problem of the parallel-plate capacitor forms the background to modern microstrip antennas in theory and practice [6].

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